# A DEEPER LOOK 11 The random walk

The aim here is to provide the details that led to eqn 16C.14 of the text:

$$P(x,t)=\left(\frac{2\tau}{\pi t}\right)^{1/2}\mathrm{e}^{-x^2\tau/2td^2} \quad (1)$$

As in the text, suppose that $N_{\mathrm{R}}$ steps are taken to the right and $N_{\mathrm{L}}$ are taken to the left (in any order), for a total number of steps $N = N_{\mathrm{L}} + N_{\mathrm{R}}$. The probability of this mixture of steps occurring is

$$P=\frac{W}{2^N}=\frac{N!}{(N-N_{\mathrm{R}})!N_{\mathrm{R}}!2^N}$$

**Step 1** *Express the probability as a logarithm and use Stirling's approximation to approximate the factorials*

The result of taking (natural) logarithms of both sides is

$$\ln P=\ln N!-\{\ln(N-N_{\mathrm{R}})!+\ln N_{\mathrm{R}}!+\ln 2^N\}$$

Now use the more accurate form of Stirling's approximation, which is

$$\ln x! \approx \ln(2\pi)^{1/2}+(x+\tfrac{1}{2})\ln x-x$$

and find

$$\begin{aligned}\ln P=&\ln(2\pi)^{1/2}+\left(N+\tfrac{1}{2}\right)\ln N-N-\ln(2\pi)^{1/2}\\&-\left(N-N_{\mathrm{R}}+\tfrac{1}{2}\right)\ln(N-N_{\mathrm{R}})+N-N_{\mathrm{R}}-\ln(2\pi)^{1/2}\\&-\left(N_{\mathrm{R}}+\tfrac{1}{2}\right)\ln N_{\mathrm{R}}+N_{\mathrm{R}}-\ln 2^N\end{aligned}$$

$$\begin{aligned}=&-\ln(2\pi)^{1/2}2^N+\left(N+\tfrac{1}{2}\right)\ln N\\&-\left(N-N_{\mathrm{R}}+\tfrac{1}{2}\right)\ln(N-N_{\mathrm{R}})-\left(N_{\mathrm{R}}+\tfrac{1}{2}\right)\ln N_{\mathrm{R}}\end{aligned}$$

$$\begin{aligned}=&-\ln(2\pi)^{1/2}2^N+\left(N+\tfrac{1}{2}\right)\ln\frac{N}{N-N_{\mathrm{R}}}\\&+N_{\mathrm{R}}\ln\frac{N-N_{\mathrm{R}}}{N_{\mathrm{R}}}-\tfrac{1}{2}\ln N_{\mathrm{R}}\end{aligned}$$

$$\begin{aligned}=&-\ln(2\pi)^{1/2}2^N+\left(N+\tfrac{1}{2}\right)\ln\frac{1}{1-N_{\mathrm{R}}/N}\\&+N_{\mathrm{R}}\ln\frac{1-N_{\mathrm{R}}/N}{N_{\mathrm{R}}/N}-\tfrac{1}{2}\ln N_{\mathrm{R}}\end{aligned}$$

**Step 2** *Expand the logarithms*

Introduce the parameter $\mu$:

$$\mu=\frac{N_{\mathrm{R}}}{N}-\tfrac{1}{2},\quad \text{so } \frac{N_{\mathrm{R}}}{N}=\mu+\tfrac{1}{2} \text{ and } N_{\mathrm{R}}=N\left(\mu+\tfrac{1}{2}\right)$$

and obtain

$$\begin{aligned}\ln P=&-\ln(2\pi)^{1/2}2^N+\left(N+\tfrac{1}{2}\right)\ln\frac{1}{\tfrac{1}{2}-\mu}\\&+N\left(\mu+\tfrac{1}{2}\right)\ln\frac{\tfrac{1}{2}-\mu}{\mu+\tfrac{1}{2}}-\tfrac{1}{2}\ln N\left(\mu+\tfrac{1}{2}\right)\end{aligned}$$

$$\begin{aligned}=&-\ln(2\pi)^{1/2}2^N-\left(N+\tfrac{1}{2}\right)\ln\left(\tfrac{1}{2}-\mu\right)+N\left(\mu+\tfrac{1}{2}\right)\ln\left(\tfrac{1}{2}-\mu\right)\\&-N\left(\mu+\tfrac{1}{2}\right)\ln\left(\mu+\tfrac{1}{2}\right)-\tfrac{1}{2}\ln N\left(\mu+\tfrac{1}{2}\right)\end{aligned}$$

Then, because it is likely that there are approximately the same numbers of steps to the left and the right, $N_{\mathrm{R}}\approx\tfrac{1}{2}N$, and $\mu$ is close to 0. Therefore the expansion

$$\ln\left(\tfrac{1}{2}\pm\mu\right)=-\ln 2\pm 2\mu-2\mu^2+\cdots$$

can be used, and terminated at the term in $\mu^2$. The result is

$$\begin{aligned}\ln P=&-\ln(2\pi N)^{1/2}2^N-\left(N+\tfrac{1}{2}\right)\{-\ln 2-2\mu-2\mu^2\}\\&+N\left(\mu+\tfrac{1}{2}\right)\{-\ln 2-2\mu-2\mu^2\}\\&-N\left(\mu+\tfrac{1}{2}\right)\{-\ln 2+2\mu-2\mu^2\}-\tfrac{1}{2}\{-\ln 2+2\mu-2\mu^2\}\end{aligned}$$

$$\begin{aligned}=&-\ln(2\pi N)^{1/2}2^N-\left(N+\tfrac{1}{2}\right)\{-\ln 2-2\mu-2\mu^2\}\\&-4N\left(\mu+\tfrac{1}{2}\right)\mu-\tfrac{1}{2}\{-\ln 2+2\mu-2\mu^2\}\end{aligned}$$

$$\begin{aligned}=&-\ln(2\pi N)^{1/2}2^N-N\{-\ln 2-2\mu-2\mu^2\}\\&-4N\left(\mu+\tfrac{1}{2}\right)\mu+\{\ln 2+2\mu^2\}\end{aligned}$$

$$=-\ln(2\pi N)^{1/2}2^N+(N+1)\ln 2-2(N-1)\mu^2$$

**Step 3** *Take antilogarithms*

Finally, take (natural) antilogarithms of both sides and use the fact that $N \gg 1$:

$$P=\frac{2^{N+1}\mathrm{e}^{-2(N-1)\mu^2}}{2^N(2\pi N)^{1/2}}=\frac{2\mathrm{e}^{-2(N-1)\mu^2}}{(2\pi N)^{1/2}}\approx\frac{2\mathrm{e}^{-2N\mu^2}}{(2\pi N)^{1/2}}$$

With $N=t/\tau$ and $N\mu^2=n^2/4N=\tau x^2/4td^2$, eqn 16C.14 now follows.