THE CHEMIST'S TOOLKIT 14 Complex numbers

Complex numbers have the general form

$$z = x + \mathrm{i}y \tag{14.1a}$$

where $\mathrm{i}=\sqrt{-1}$. The real number x is the 'real part of z', denoted Re(z); likewise, the real number y is 'the imaginary part of z', Im(z). The **complex conjugate** of z, denoted z^*, is formed by replacing i by −i:

$$z^* = x - \mathrm{i}y \tag{14.1b}$$

Brief illustration 14.1: Operations with complex numbers

Consider the complex numbers $z_1 = 6 + 2\mathrm{i}$ and $z_2 = -4 - 3\mathrm{i}$. Then

$$z_1 + z_2 = (6-4) + (2-3)\mathrm{i} = 2 - \mathrm{i}$$
$$z_1 - z_2 = 10 + 5\mathrm{i}$$
$$z_1 z_2 = \{6(-4) - 2(-3)\} + \{6(-3) + 2(-4)\}\mathrm{i} = -18 - 26\mathrm{i}$$

The product of z^* and z is denoted $|z|^2$ and is called the **square modulus** of z. From the definition of z and z^* and $\mathrm{i}^2 = -1$ it follows that

$$|z|^2 = z^*z = (x+\mathrm{i}y)(x-\mathrm{i}y) = x^2 + y^2 \tag{14.2}$$

The square modulus is a real, non-negative number. The **absolute value** or **modulus** is denoted $|z|$ and is given by:

$$|z| = (z^*z)^{1/2} = (x^2+y^2)^{1/2} \tag{14.3}$$

The inverse of z, denoted z^{-1}, is such that $zz^{-1} = 1$, which is satisfied if

$$z^{-1} = \frac{z^*}{|z|^2} \tag{14.4}$$

This construction is used in the division of complex numbers:

$$\frac{z_1}{z_2} = z_1 z_2^{-1} \tag{14.5}$$

Brief illustration 14.2: Inverse

Consider the complex number $z = 8 - 3\mathrm{i}$. Its square modulus is

$$|z|^2 = z^*z = (8-3\mathrm{i})^*(8-3\mathrm{i}) = (8+3\mathrm{i})(8-3\mathrm{i}) = 64 + 9 = 73$$

The modulus is therefore $|z| = 73^{1/2}$. The inverse of z is

$$z^{-1} = \frac{8+3\mathrm{i}}{73} = \frac{8}{73} + \frac{3}{73}\mathrm{i}$$

Then

$$\frac{6+2\mathrm{i}}{8-3\mathrm{i}} = (6+2\mathrm{i})(8-3\mathrm{i})^{-1} = (6+2\mathrm{i})\left(\frac{8}{73} + \frac{3}{73}\mathrm{i}\right) = -\frac{42}{73} + \frac{34}{73}\mathrm{i}$$

For further information about complex numbers, see *The chemist's toolkit* 16 .