2 Curves and graphs

(1) Find the equation of the parabola that passes through the three points $(0,3)$, $(3,0)$, and $(5,8)$; what are the roots of the equation?

General equation of a parabola is $y = ax^2 + bx + c$.

$$\left.\begin{array}{lll} \text{Passes through } (0,3) & \Rightarrow & 3 = c \\ \text{Passes through } (3,0) & \Rightarrow & 0 = 9a + 3b + c \\ \text{Passes through } (5,8) & \Rightarrow & 8 = 25a + 5b + c \end{array}\right\} \quad \begin{array}{l} c = 3 \\ a = 1 \\ b = -4 \end{array}$$

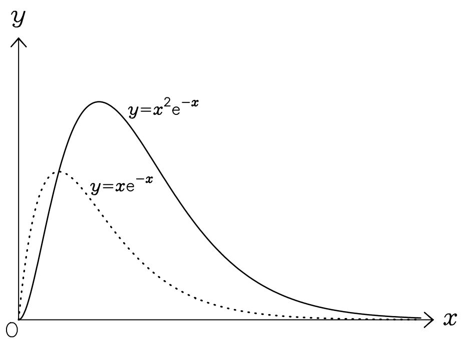
$\therefore$ Equation of parabola is $\underline{y = x^2 - 4x + 3}$

For roots, $y = 0 \;\Rightarrow\; x^2 - 4x + 3 = (x-3)(x-1) = 0$

i.e. $\underline{x = 1 \text{ and } x = 3}$

(2) Given that the exponential dominates for large x, sketch the functions $y = x\,e^{-x}$ and $y = x^2\,e^{-x}$ for $x \geqslant 0$.

Close to the origin, $x \ll 1$, $e^{-x} \approx 1$, and so the shape of the curves is determined by the exponential prefactors: a linear rise for $x\,e^{-x}$, and a parabolic increase for $x^2\,e^{-x}$. For large x, both curves decay to zero.



(3) What is the centre and radius of the circle $x^2-2x+y^2+4y-4=0$?

$$x^2-2x+y^2+4y = 4$$
$$\therefore\quad (x-1)^2-1+(y+2)^2-4 = 4$$
$$\therefore\quad (x-1)^2+(y+2)^2 = 9 = 3^2$$

i.e. centre is at $(1,-2)$ and radius $=3$

This 'completing the square' transformation of the original equation into the form $(x-x_0)^2+(y-y_0)^2=r^2$ is useful because it then becomes obvious that we have a circle with centre (x_0,y_0) and radius r. Alternatively, we could remember that $x^2+y^2+2gx+2hy+c=0$ is a general equation for a circle; its centre is at $(-g,-h)$ and the radius is $\sqrt{g^2+h^2-c}$.

(4) Sketch the ellipse $3x^2+4y^2=3$; evaluate its eccentricity, and indicate the positions of the focus and directrix.

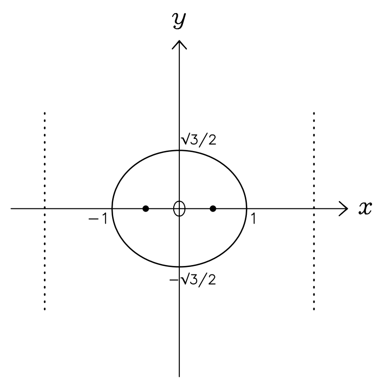
The simplest form for the equation of an ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$$

where the centre is at the origin and the principal axes are aligned with the x and y axes, and of width $2a$ and $2b$ respectively. This is the case for

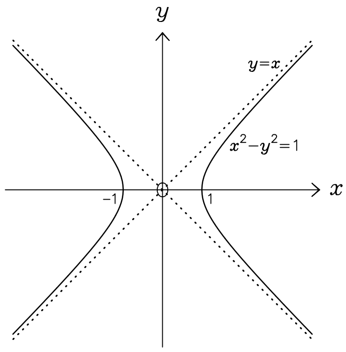
$$x^2+\frac{4}{3}y^2 = 1$$

where $a=1$ and $b=\sqrt{3}/2$.



The eccentricity ϵ is related to a and b through $b^2=a^2(1-\epsilon^2)$, giving $\epsilon=1/2$. The foci are at $(\pm a\epsilon,0)$, or $(\pm 1/2,0)$, and the two directrices are the straight lines $x=\pm a/\epsilon=\pm 2$.

(5) Sketch the hyperbola $x^2 - y^2 = 1$, and mark in the asymptotes.



(6) By first factorizing the equation, or otherwise, sketch the function $(x^2+y^2)^2 - 4x^2 = 0$.

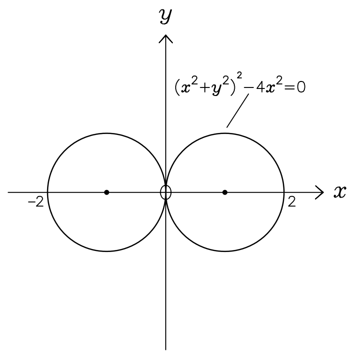
The expression is of the form $a^2 - b^2$, and can be factorized as $(a-b)(a+b)$:

$$(x^2+y^2-2x)(x^2+y^2+2x) = 0$$

$$\therefore \quad x^2-2x+y^2 = 0 \quad \text{or} \quad x^2+2x+y^2 = 0$$

$$\text{i.e.} \quad (x-1)^2+y^2 = 1 \quad \text{or} \quad (x+1)^2+y^2 = 1$$

Two circles of radius 1, with centres at $(1,0)$ and $(-1,0)$.



If the factorization hint had not been given, or spotted, then an obvious way of reducing the complexity of the given equation would be to write it as

$$(x^2+y^2)^2 = 4x^2$$

and take the square root of both sides. This gives

$$x^2+y^2 = \pm 2x$$

which also leads to the equation of the two circles above.