9 Matrices

(1) Show how the determinant of a 3×3 matrix can be used to evaluate both a cross product and a scalar triple product.

Let the vectors **a**, **b** and **c** have components

$$\mathbf{a} = (a_1, a_2, a_3) = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$

$$\mathbf{b} = (b_1, b_2, b_3) = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$$

$$\mathbf{c} = (c_1, c_2, c_3) = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$$

where **i**, **j** and **k** are unit vectors in the x, y and z directions respectively. Then

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2 b_3 - b_2 a_3)\,\mathbf{i} - (a_1 b_3 - b_1 a_3)\,\mathbf{j} + (a_1 b_2 - b_1 a_2)\,\mathbf{k}$$

$$= \big(a_2 b_3 - b_2 a_3\,,\; a_3 b_1 - b_3 a_1\,,\; a_1 b_2 - b_1 a_2\big) = \underline{\mathbf{a}\times\mathbf{b}}$$

$$\begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2 b_3 - b_2 a_3)\,c_1 + (a_3 b_1 - b_3 a_1)\,c_2 + (a_1 b_2 - b_1 a_2)\,c_3$$

$$= (\mathbf{a}\times\mathbf{b})_1\, c_1 + (\mathbf{a}\times\mathbf{b})_2\, c_2 + (\mathbf{a}\times\mathbf{b})_3\, c_3 = \underline{(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}}$$

The general properties of determinants can be used to infer some of those of a cross and scalar triple product:

(i) $\mathbf{a}\times\mathbf{b} = -\,\mathbf{b}\times\mathbf{a}$, because the determinant is multiplied by -1 if two rows (or columns) are interchanged;

(ii) $\mathbf{a}\times\mathbf{b} = 0$ if **a** is parallel to **b**, because a determinant is zero if two rows are the same;

(iii) $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = 0$ if **a**, **b** and **c** are not linearly independent, so that they lie in a plane (or along a line), for then any row minus a suitable combination of the other two will result in a whole line of zeros.

(2) Find the inverse of the following matrix

$$\mathbb{C} = \begin{pmatrix} 2 & -1 & 1 \\ 1 & -1 & 2 \\ -1 & 1 & -1 \end{pmatrix}$$

and verify that $\mathbb{C}\,\mathbb{C}^{-1} = \mathbb{C}^{-1}\mathbb{C} = \mathbb{I}$.

$\text{adj}(\mathbb{C})$ = matrix of cofactors of $\mathbb{C}^{\text{T}}$

$$\mathbb{C}^{\text{T}} = \begin{pmatrix} 2 & 1 & -1 \\ -1 & -1 & 1 \\ 1 & 2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} (-1)\times(-1) - 2\times 1 & 1\times 1 - (-1)\times(-1) & (-1)\times 2 - 1\times(-1) \\ 2\times(-1) - 1\times(-1) & 2\times(-1) - 1\times(-1) & 1\times 1 - 2\times 2 \\ 1\times 1 - (-1)\times(-1) & (-1)\times(-1) - 2\times 1 & 2\times(-1) - (-1)\times 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 & -1 \\ -1 & -1 & -3 \\ 0 & -1 & -1 \end{pmatrix}$$

$$\det(\mathbb{C}) = \det(\mathbb{C}^{\text{T}})$$

= dot product of any row, or column, with its cofactors

$$= 2\times(-1) + 1\times 0 + (-1)\times(-1) = -1$$

$$\therefore\quad \mathbb{C}^{-1} = \frac{\text{adj}(\mathbb{C})}{\det(\mathbb{C})} = \underline{\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 3 \\ 0 & 1 & 1 \end{pmatrix}}$$

$$\mathbb{C}\,\mathbb{C}^{-1} = \begin{pmatrix} 2-1+0 & 0-1+1 & 2-3+1 \\ 1-1+0 & 0-1+2 & 1-3+2 \\ -1+1+0 & 0+1-1 & -1+3-1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbb{I}$$

$$\mathbb{C}^{-1}\mathbb{C} = \begin{pmatrix} 2+0-1 & -1+0+1 & 1+0-1 \\ 2+1-3 & -1-1+3 & 1+2-3 \\ 0+1-1 & 0-1+1 & 0+2-1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbb{I}$$